MATH 2230 Complex Variables with Applications (2014-2015, Term 1) Solution and Remarks to Midterm 1

1. (1) ab
   (2) c
   (3) c
   (4) acd
   (5) acd
   (6) ab

   Remark:
   For single choice questions, you will obtain marks only when you choose the correct answer.
   For multiple choice questions, there are three cases:
   (1) You will obtain full marks when you choose all correct answers.
   (2) You will obtain part of the scores when you choose only some of the correct answers.
   For question 1 and 6, you will be marked 2 points when you choose one correct answer.
   For question 4 and 5, you will be marked 1 point or 3 points respectively when you choose one or two correct answers.
   (3) You will be marked 0 whenever you choose one incorrect answer.

2. Solution:

$$R(z) = \frac{z^5}{(z^2+1)^2} = z + \frac{-2z^3 - z}{z^4 + 2z^2 + 1}$$

Let

$$\frac{-2z^3 - z}{(z^2+1)^2} = \frac{A}{z+i} + \frac{B}{z-i} + \frac{C}{(z+i)^2} + \frac{D}{(z-i)^2}$$

Then

$$\frac{-2z^3 - z}{(z^2+1)^2} = \frac{A(z-i)^2(z+i) + B(z+i)^2(z-i) + C(z-i)^2 + D(z+i)^2}{(z^2+1)^2}$$

$$\frac{-2z^3 - z}{(z^2+1)^2} = \frac{A(z^3 - iz^2 + z - i) + B(z^3 + iz^2 + z + i) + C(z^2 - 2iz - 1) + D(z^2 + 2iz - 1)}{(z^2+1)^2}$$

$$\begin{cases} A + B = -2 \\ -iA + iB + C + D = 0 \\ A + B - 2iC + 2iD = -1 \\ -Ai + Bi - C - D = 0 \end{cases}$$

Thus, $A = -1, B = -1, C = \frac{i}{4}, D = -\frac{i}{4}$.
Therefore,

$$R(z) = z - \frac{1}{z+i} - \frac{1}{z-i} + \frac{i}{4(z+i)^2} - \frac{i}{4(z-i)^2}$$

Remark:If you have the correct method but some mistakes in computation, I will only deduct a few points.

3. Solution: Suppose $v(x,y)$ is a harmonic conjugate of u. Then

$$u_x = v_y = -\sin x \cosh y$$

$$u_y = -v_x = \cos x \sinh y$$

By the first equation, we have $v(x,y) = -\sin x \sinh y + C(x)$, where $C(x)$ is a function of x.
Then $v_x = -\cos x \sinh y + C'(x) = -\cos x \sinh y$ (by the second equation)
Thus, $C(x) = C$, where C is a constant.
Therefore,

$$v(x,y) = -\sin x \sinh y + C$$

where C is a constant
Noted that the two functions u and v are harmonic in $\mathbb{C}$ and their first-order partial derivatives satisfy the Cauchy-Riemann equations

$$u_x = v_y, u_y = -v_x$$

throughout $\mathbb{C}$.
Thus v is indeed a harmonic conjugate of u.
The derivative of $u + iv$ is

$$u_x + iv_x = -\sin x \cosh y - i \cos x \sinh y$$

Remark: Finding v values 20 points and calculating the derivative of $u + iv$ values 5 points.
If you have the correct method but some mistakes in computation, I will only deduct a few points.

4. Solution:

(a) Suppose the center of the circle C is $a = x + iy$.
Then we have

$$6 = a + \frac{4^2}{\bar{0} - \bar{a}}$$

$$6 = x + iy - \frac{16}{x - iy}$$

We get $y = 0$, $x = -2$ or $x = 8$.
Thus, the collection of the circle C are

$$|z + 2| = 4$$

and

$$|z - 8| = 4$$

(b) Obviously, C_1 is $|z + 2| = 4$.
Since $(0, \infty)$ is a symmetric pair of the circle $|z| = 1$, we have $(0, T\infty)$ is a symmetric pair of C_1.
Then

$$T\infty = -2 + \frac{4^2}{\bar{0} - \overline{-2}} = 6$$

Thus,

$$(z, 0, 1, \infty) = (T(z), 0, 2, 6)$$

$$T(z) = \frac{6z}{2 + z}$$

(c) Noted that 0 is a point in the interior of $|z| = 1$ and $T(0)$ is also in the interior of C_1.
Thus, the exterior of $|z| = 1$ is mapped to the exterior of C_1.